Application of a Bernstein type inequality to rational interpolation in the Dirichlet space

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APPLICATION OF A BERNSTEIN TYPE INEQUALITY TO RATIONAL INTERPOLATION IN THE DIRICHLET SPACE

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Abstract. We prove a Bernstein-type inequality involving the Bergman and the Hardy norms, for rational functions in the unit disc $\mathbb{D}$ having at most n poles all outside of $\frac{1}{r}\mathbb{D}$, $0<r<1$. The asymptotic sharpness of this inequality is shown as $n\to\infty$ and $r\to 1^{-}$. We apply our Bernstein-type inequality to an effective Nevanlinna-Pick interpolation problem in the standard Dirichlet space, constrained by the H^2- norm.

Introduction

a. Statement of the problems.

Let $\mathbb{D}=\{z\in\mathbb{C}:\,|z|<1\}$ be the unit disc of the complex plane and let $\mathrm{Hol}\,(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$. Let also X and Y be two Banach spaces of holomorphic functions on the unit disc $\mathbb{D}$, $X,\,Y\subset\mathrm{Hol}\,(\mathbb{D})$. Here and later on, H^∞ stands for the space (algebra) of bounded holomorphic functions in the unit disc $\mathbb{D}$ endowed with the norm $\|f\|_\infty=\sup_{z\in\mathbb{D}}|f(z)|$. We suppose that $n\geq 1$ is an integer, $r\in[0,\,1)$ and we consider the two following problems.

Problem 1. Let $\mathcal{P}_n$ be the complex space of analytic polynomials of degree less or equal than n, and

$$\mathcal{R}_{n,\,r}=\left\{\frac{p}{q}\,:\;q\in\mathcal{P}_n,\;d^\circ p<d^\circ q,\;q(\zeta)=0\Longrightarrow|\zeta|\geq\frac{1}{r}\right\},$$

(where $d^\circ p$ means the degree of any $p\in\mathcal{P}_n$) be the set of all rational functions in $\mathbb{D}$ of degree less or equal than $n\geq 1$, having at most n poles all outside of $\frac{1}{r}\mathbb{D}$. Notice that for $r=0$, we get $\mathcal{R}_{n,\,0}=\mathcal{P}_{n-1}$. Our first problem is to search for the "best possible" constant $\mathcal{C}_{n,\,r}(X,\,Y)$ such that

$$\|f'\|_X\leq\mathcal{C}_{n,\,r}(X,\,Y)\,\|f\|_Y$$

for all $f\in\mathcal{R}_{n,\,r}$.

Problem 2. Let $\sigma=\{\lambda_1,...,\lambda_n\}$ be a finite subset of $\mathbb{D}$. What is the best possible interpolation by functions of the space Y for the traces $f_{|\sigma}$ of functions of the space X, in the worst case? The case $X\subset Y$ is of no interest, and so one can suppose that either $Y\subset X$ or X and Y are incomparable. More precisely, our second problem is to compute or estimate the following interpolation constant

$$I\,(\sigma,\,X,\,Y)=\sup_{f\in X,\,\|f\|_X\leq 1}\inf\left\{\|g\|_Y:\,g_{|\sigma}=f_{|\sigma}\right\}.$$

We also define

$$\mathcal{I}_{n,r}(X,\,Y)=\sup\left\{I(\sigma,\,X,\,Y)\,:\,\mathrm{card}\,\sigma\leq n\,,\,|\lambda|\leq r,\,\forall\lambda\in\sigma\right\}.$$

b. Motivations.

Problem 1. Bernstein-type inequalities for rational functions are applied

1.1. in matrix analysis and in operator theory (see “Kreiss Matrix Theorem” [LeTr, Sp] or [Z1, Z4] for resolvent estimates of power bounded matrices),

1.2. to “inverse theorems of rational approximation” using the *classical Bernstein decomposition* (see [Da, Pel, Pek]),

1.3. to effective H^∞ interpolation problems (see [Z3] and our Theorem B below in Subsection d), and more generally to our Problem 1.

Problem 2. We can give three main motivations for Problem 2.

2.1. It is explained in [Z3] (the case $Y=H^\infty$) why the classical interpolation problems, those of Nevanlinna-Pick (1908) and Carathéodory-Schur (1916) (see [N2] p.231 for these two problems), on the one hand and Carleson's free interpolation problem (1958) (see [N1] p.158) on the other hand, are of the nature of our interpolation problem.

2.2. It is also explained in [Z3] why this constrained interpolation is motivated by some applications in matrix analysis and in operator theory.

2.3. It has already been proved in [Z3] that for $X=H^2$ (see Subsection c. for the definition of H^2) and $Y=H^\infty$,

$$\frac{1}{4\sqrt{2}}\frac{\sqrt{n}}{\sqrt{1-r}}\leq\mathcal{I}_{n,r}\left(H^2,\,H^\infty\right)\leq\sqrt{2}\frac{\sqrt{n}}{\sqrt{1-r}}.\tag{1}$$

The above estimate (1) answers a question of L. Baratchart (private communication), which is part of a more complicated question arising in an applied situation in [BL1] and [BL2]: given a set $\sigma\subset\mathbb{D}$, how to estimate $I\left(\sigma,\,H^2,\,H^\infty\right)$ in terms of $n=\mathrm{card}(\sigma)$ and $\max_{\lambda\in\sigma}|\lambda|=r$ only?

c. The spaces X and Y considered here.

Now let us define some Banach spaces X and Y of holomorphic functions in $\mathbb{D}$ which we will consider throughout this paper. From now on, if $f\in\mathrm{Hol}(\mathbb{D})$ and $k\in\mathbb{N}$, $\hat{f}(k)$ stands for the k^{th} Taylor coefficient of f.

1. The standard Hardy space $H^2=H^2(\mathbb{D})$,

$$H^2=\left\{f\in\mathrm{Hol}\left(\mathbb{D}\right):\;\|f\|_{H^2}^2=\sup_{0\leq r<1}\int_{\mathbb{T}}|f(rz)|^2\,\mathrm{d}m(z)<\infty\right\},$$

where m stands for the normalized Lebesgue measure on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. An equivalent description of the space H^2 is

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{H^2} = \left(\sum_{k \geq 0} \left| \hat{f}(k) \right|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

2. The standard Bergman space $L^2_a = L^2_a(\mathbb{D})$,

$$L^2_a = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|^2_{L^2_a} = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 \, \mathrm{d}A(z) < \infty \right\},$$

where A is the standard area measure, also defined by

$$L^2_a = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{L^2_a} = \left(\sum_{k \geq 0} \left| \hat{f}(k) \right|^2 \frac{1}{k+1} \right)^{\frac{1}{2}} < \infty \right\}.$$

3. The analytic Besov space $B^{\frac{1}{2}}_{2,\,2}$ (also known as the standard Dirichlet space) defined by

$$B^{\frac{1}{2}}_{2,\,2} = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{B^{\frac{1}{2}}_{2,\,2}} = \left(\sum_{k \geq 0} (k+1) \left| \hat{f}(k) \right|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

Then if $f \in B^{\frac{1}{2}}_{2,\,2}$, we have the following equality

$$\|f\|^2_{B^{\frac{1}{2}}_{2,\,2}} = \|f'\|^2_{L^2_a} + \|f\|^2_{H^2}, \tag{2}$$

which establishes a link between the spaces $B^{\frac{1}{2}}_{2,\,2}$ and L^2_a.

d. The results. Here and later on, the letter c denotes a positive constant that may change from one step to the next. For two positive functions a and b, we say that a is dominated by b, denoted by $a = O(b)$, if there is a constant $c > 0$ such that $a \leq cb$; and we say that a and b are comparable, denoted by $a \asymp b$, if both $a = O(b)$ and $b = O(a)$ hold.

Problem 1. Our first result (Theorem A, below) is a partial case ($p = q = 2,\ s = \frac{1}{2}$) of the following K. Dyakonov's result [Dy]: if $p \in [1, \infty),\ s \in (0, +\infty),\ q \in [1, +\infty]$, then there exists a constant $c_{p,\,s} > 0$ such that

$$\mathcal{C}_{n,\,r}\left(B^{s-1}_{p,\,p},\ H^q\right) \leq c_{p,\,s} \sup \|B'\|^s_{H^\gamma}, \tag{3}$$

where γ is such that $\frac{s}{\gamma} + \frac{1}{q} = \frac{1}{p}$, and the supremum is taken over all finite Blaschke products B of order n with n zeros outside of $\frac{1}{r}\mathbb{D}$. Here $B^s_{p,\,p}$ stands for the Hardy-Besov space which consists of analytic functions f on $\mathbb{D}$ satisfying

$$\|f\|_{B^s_{p,\,p}} = \sum_{k=0}^{n-1} \left|f^{(k)}(0)\right| + \int_{\mathbb{D}} (1-|w|)^{(n-s)p-1} \left|f^{(n)}(w)\right|^p \mathrm{d}A(w) < \infty.$$

For the (tiny) partial case considered here, our proof is different and the constant $c_{2,\,\frac{1}{2}}$ is asymptotically sharp as r tends to 1^- and n tends to $+\infty$.

Theorem A. *Let $n \geq 1$ and $r \in [0,\,1)$. We have*

(i)

$$\widetilde{a}(n,\,r)\sqrt{\frac{n}{1-r}} \leq \mathcal{C}_{n,\,r}\left(L^2_a,\,H^2\right) \leq \widetilde{A}(n,\,r)\sqrt{\frac{n}{1-r}}, \tag{4}$$

where

$$\widetilde{a}(n,\,r) \geq \left(1-\frac{1-r}{n}\right)^{\frac{1}{2}} \text{ and } \widetilde{A}(n,\,r) \leq \left(1+r+\frac{1}{\sqrt{n}}\right)^{\frac{1}{2}}.$$

(ii) Moreover, the sequence

$$\left(\frac{\mathcal{C}_{n,\,r}\left(L^2_a,\,H^2\right)}{\sqrt{n}}\right)_{n\geq 1}$$

is convergent and there exists a limit

$$\lim_{n\to\infty} \frac{\mathcal{C}_{n,\,r}\left(L^2_a,\,H^2\right)}{\sqrt{n}} = \sqrt{\frac{1+r}{1-r}}. \tag{5}$$

for all $r \in [0,\,1)$.

Notice that it has already been proved in [Z2] that there exists a limit

$$\lim_{n\to\infty} \frac{\mathcal{C}_{n,\,r}\left(H^2,\,H^2\right)}{n} = \frac{1+r}{1-r}, \tag{6}$$

for every $r,\ 0 \leq r < 1$.

Problem 2. Looking at motivation 2.3, we replace the algebra H^∞ by the Dirichlet space $B^{\frac{1}{2}}_{2,\,2}$. We show that the "gap" between $X = H^2$ and $Y = H^\infty$ (see (1)) is asymptotically the same as the one which exists between $X = H^2$ and $Y = B^{\frac{1}{2}}_{2,\,2}$. In other words,

$$\mathcal{I}_{n,\,r}\left(H^2,\,B^{\frac{1}{2}}_{2,\,2}\right) \asymp \mathcal{I}_{n,\,r}\left(H^2,\,H^\infty\right) \asymp \sqrt{\frac{n}{1-r}}. \tag{7}$$

More precisely, we prove the following Theorem B, in which the right-hand side inequality of (10) is a consequence of the right-hand side inequality of (4) in the above Theorem A.

Theorem B. *Let $n \geq 1$, and $r \in [0, 1)$. Then,*

$$\mathcal{I}_{n,r}\left(H^2,\, B^{\frac{1}{2}}_{2,2}\right) \leq \left[\left(\mathcal{C}_{n,r}\left(L^2_a,\, H^2\right)\right)^2 + 1\right]^{\frac{1}{2}}. \tag{8}$$

Let $\lambda \in \mathbb{D}$ and the corresponding one-point interpolation set $\sigma_{n,\lambda} = \underbrace{\{\lambda, \lambda, ..., \lambda\}}_{n}$. We have,

$$I\left(\sigma_{n,\lambda}, H^2,\, B^{\frac{1}{2}}_{2,2}\right) \geq \sqrt{\frac{n}{1-|\lambda|}}\left[\frac{(1+|\lambda|)^2 - \frac{2}{n} - \frac{2|\lambda|}{n}}{2(1+|\lambda|)}\right]^{\frac{1}{2}}. \tag{9}$$

In particular,

$$\left[\frac{1+r}{2}\left(1-\frac{1}{n}\right)\right]^{\frac{1}{2}}\sqrt{\frac{n}{1-r}} \leq \mathcal{I}_{n,r}\left(H^2,\, B^{\frac{1}{2}}_{2,2}\right) \leq \left(1 + r + \frac{1}{\sqrt{n}} + \frac{1-r}{n}\right)^{\frac{1}{2}}\sqrt{\frac{n}{1-r}}, \tag{10}$$

$$\sqrt{\frac{\frac{1+r}{2}}{1-r}} \leq \liminf_{n\to\infty}\frac{\mathcal{I}_{n,r}\left(H^2,\, B^{\frac{1}{2}}_{2,2}\right)}{\sqrt{n}} \leq \limsup_{n\to\infty}\frac{\mathcal{I}_{n,r}\left(H^2,\, B^{\frac{1}{2}}_{2,2}\right)}{\sqrt{n}} \leq \sqrt{\frac{1+r}{1-r}}, \tag{11}$$

and

$$\frac{\sqrt{2}}{2} \leq \liminf_{r\to1^-}\liminf_{n\to\infty}\sqrt{\frac{1-r}{n}}\mathcal{I}_{n,r}\left(H^2,\, B^{\frac{1}{2}}_{2,2}\right) \leq \limsup_{r\to1^-}\limsup_{n\to\infty}\sqrt{\frac{1-r}{n}}\mathcal{I}_{n,r}\left(H^2,\, B^{\frac{1}{2}}_{2,2}\right) \leq \sqrt{2}. \tag{12}$$

In the next Section, we first give some definitions introducing the main tools used in the proofs of Theorem A and Theorem B. After that, we prove these theorems.

Proofs of Theorems A and B

From now on, if $\sigma = \{\lambda_1, ..., \lambda_n\} \subset \mathbb{D}$ is a finite subset of the unit disc, then

$$B_\sigma = \prod_{j=1}^{n} b_{\lambda_j}$$

is the corresponding finite Blaschke product where $b_\lambda = \frac{\lambda - z}{1-\overline{\lambda}z}$, $\lambda \in \mathbb{D}$. In Definitions 1, 2, 3 and in Remark 4 below, $\sigma = \{\lambda_1,\, ...,\, \lambda_n\}$ is a sequence in the unit disc $\mathbb{D}$ and B_σ is the corresponding Blaschke product.

Definition 1. *Malmquist family.* For $k \in [1,\, n]$, we set $f_k = \frac{1}{1-\overline{\lambda_k}z}$, and define the family $(e_k)_{1\leq k\leq n}$, (which is known as Malmquist basis, see [N1, p.117]), by

$$e_1 = \frac{f_1}{\|f_1\|_2} \text{ and } e_k = \left(\prod_{j=1}^{k-1} b_{\lambda_j}\right) \frac{f_k}{\|f_k\|_2}, \tag{13}$$

for $k \in [2,\, n]$; we have $\|f_k\|_2 = \left(1 - |\lambda_k|^2\right)^{-1/2}$.

Definition 2. *The model space K_{B_σ}.* We define K_{B_σ} to be the n-dimensional space:

$$K_{B_\sigma} = \left(B_\sigma H^2\right)^\perp = H^2 \ominus B_\sigma H^2. \tag{14}$$

Definition 3. *The orthogonal projection P_{B_σ} on K_{B_σ}.* We define P_{B_σ} to be the orthogonal projection of H^2 on its n-dimensional subspace K_{B_σ}.

Remark 4. The Malmquist family $(e_k)_{1\leq k\leq n}$ corresponding to σ is an orthonormal basis of K_{B_σ}. In particular,

$$P_{B_\sigma} = \sum_{k=1}^{n} \left(\cdot,\, e_k\right)_{H^2} e_k, \tag{15}$$

where $(\cdot,\, \cdot)_{H^2}$ means the scalar product on H^2.

Proof of Theorem A.

Proof of (i). 1) We fist prove the the right-hand side inequality of (4). Using both Cauchy-Schwarz inequality and the fact that $\widehat{f'}(k) = (k+1)\widehat{f}(k+1)$ for all $k \geq 0$, we get

$$\|f'\|^2_{L^2_a} = \sum_{k\geq 0} \frac{\left|\widehat{f'}(k)\right|^2}{k+1} = \sum_{k\geq 0} \frac{(k+1)^2 \left|\widehat{f}(k+1)\right|^2}{k+1} =$$

$$= \sum_{k\geq 1} k \left|\widehat{f}(k)\right|^2 \leq \left(\sum_{k\geq 1} k^2 \left|\widehat{f}(k)\right|^2\right)^{\frac{1}{2}} \left(\sum_{k\geq 1} \left|\widehat{f}(k)\right|^2\right)^{\frac{1}{2}} =$$

$$= \|f'\|_{H^2} \|f\|_{H^2} \leq \mathcal{C}_{n,\,r}\left(H^2,\, H^2\right) \|f\|^2_{H^2},$$

and hence,

$$\|f'\|_{L^2_a} \leq \sqrt{\mathcal{C}_{n,\,r}\left(H^2,\, H^2\right)}\, \|f\|_{H^2},$$

which means

$$\mathcal{C}_{n,\,r}\left(L^2_a,\,H^2\right)\leq\sqrt{\mathcal{C}_{n,\,r}\left(H^2,\,H^2\right)}.$$

Then it remains to use [Z2, p.2]:

$$\mathcal{C}_{n,\,r}\left(H^2,\,H^2\right)\leq\left(1+r+\frac{1}{\sqrt{n}}\right)\frac{n}{1-r},$$

for all $n\geq 1$ and $r\in[0,\,1)$.

2) The proof of the left-hand side inequality of (4) repeates the one of [Z2, (i)] (for the left-hand side inequality) excepted that this time, we replace the Hardy norm $\|\cdot\|_{H^2}$ by the Bergman one $\|\cdot\|_{L^2_a}$. Indeed, we use the same test function $e_n=\frac{\left(1-r^2\right)^{\frac{1}{2}}}{1-rz}b_r^{n-1}$ (the n^{th} vector of the Malmquist family associated with the one-point set $\sigma_{n,\,r}=\underbrace{\{r,\,r,...,\,r\}}_{n}$ see Definition 1) and prove by the same changing of variable $\circ b_r$ (in the integral on the unit disc $\mathbb{D}$ which defines the L^2_a-norm) that

$$\|e_n'\|^2_{L^2_a}=\frac{n}{1-r}\left(1-\frac{1-r}{n}\right),$$

which gives

$$\mathcal{C}_{n,\,r}\left(L^2_a,\,H^2\right)\geq\sqrt{\frac{n}{1-r}}\left(1-\frac{1-r}{n}\right)^{\frac{1}{2}}.$$

Here are the details of the proof. We have $e_n\in K_{b_r^n}$ and $\|e_n\|_{H^2}=1$, (see [N1], Malmquist-Walsh Lemma, p.116). Moreover,

$$e_n'=\frac{r\left(1-r^2\right)^{\frac{1}{2}}}{\left(1-rz\right)^2}b_r^{n-1}+(n-1)\frac{\left(1-r^2\right)^{\frac{1}{2}}}{1-rz}b_r'b_r^{n-2}=$$

$$=-\frac{r}{\left(1-r^2\right)^{\frac{1}{2}}}b_r'b_r^{n-1}+(n-1)\frac{\left(1-r^2\right)^{\frac{1}{2}}}{1-rz}b_r'b_r^{n-2},$$

since $b_r'=\frac{r^2-1}{(1-rz)^2}$. Then,

$$e_n'=b_r'\left[-\frac{r}{\left(1-r^2\right)^{\frac{1}{2}}}b_r^{n-1}+(n-1)\frac{\left(1-r^2\right)^{\frac{1}{2}}}{1-rz}b_r^{n-2}\right],$$

and

$$\|e_n'\|^2_{L^2_a}=\frac{1}{2\pi}\int_{\mathbb{D}}|b_r'(w)|^2\left|-\frac{r}{\left(1-r^2\right)^{\frac{1}{2}}}\left(b_r(w)\right)^{n-1}+(n-1)\frac{\left(1-r^2\right)^{\frac{1}{2}}}{1-rw}\left(b_r(w)\right)^{n-2}\right|^2\mathrm{d}m(w)=$$

$$= \frac{1}{2\pi}\int_{\mathbb{D}} |b_r'(w)|^2 \left|(b_r(w))^{n-2}\right|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r(w) + (n-1)\frac{(1-r^2)^{\frac{1}{2}}}{1-rw} \right| \mathrm{d}m(w),$$

which gives, using the variables $u = b_r(w)$,

$$\|e_n'\|^2_{L^2_a} = \frac{1}{2\pi}\int_{\mathbb{D}} \left|u^{n-2}\right|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1)\frac{(1-r^2)^{\frac{1}{2}}}{1-rb_r(u)} \right|^2 \mathrm{d}m(u).$$

But $1-rb_r = \frac{1-rz-r(r-z)}{1-rz} = \frac{1-r^2}{1-rz}$ and $b_r' \circ b_r = \frac{r^2-1}{(1-rb_r)^2} = -\frac{(1-rz)^2}{1-r^2}$. This implies

$$\|e_n'\|^2_{L^2_a} = \frac{1}{2\pi}\int_{\mathbb{D}} \left|u^{n-2}\right|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1)\frac{(1-r^2)^{\frac{1}{2}}}{1-r^2}(1-ru) \right|^2 \mathrm{d}m(u) =$$

$$= \frac{1}{(1-r^2)}\frac{1}{2\pi}\int_{\mathbb{D}} \left|u^{n-2}\right|^2 \left|(-ru + (n-1)(1-ru))\right|^2 \mathrm{d}m(u),$$

which gives

$$\|e_n'\|_{L^2_a} = \frac{1}{(1-r^2)^{\frac{1}{2}}} \|\varphi_n\|_2 ,$$

where $\varphi_n = z^{n-2}\left(-rz + (n-1)(1-rz)\right)$. Expanding, we get

$$\varphi_n = z^{n-2}\left(-rz + n - 1 + rz - nrz\right) =$$

$$= z^{n-2}\left(-nrz + n - 1\right) = (n-1)z^{n-2} - nrz^{n-1},$$

and

$$\|e_n'\|^2_{L^2_a} = \frac{1}{(1-r^2)}\left(\frac{(n-1)^2}{n-1} + \frac{n^2}{n}r^2\right) = \frac{1}{(1-r^2)}\left(n(1+r) - 1\right)$$

$$= \frac{n}{(1-r)(1+r)}\left((1+r) - \frac{1}{n}\right) = \frac{n}{1-r}\left(1 - \frac{1-r}{n}\right),$$

which gives

$$\mathcal{C}_{n,r}\left(L^2_a, H^2\right) \geq \sqrt{\frac{n}{1-r}}\left(1 - \frac{1-r}{n}\right)^{\frac{1}{2}}.$$

Proof of (ii). This is again the same proof as [Z2, (ii)] (the three steps). More precisely in Step 2, we use the same test function

$$f = \sum_{k=0}^{s+2} (-1)^k e_{n-k},$$

(where $s = (s_n)$ is defined in [Z2, p.8]), and the same changing of variable $\circ b_r$ in the integral on $\mathbb{D}$. Here are the details of the proof.

Step 1. We first prove the right-hand-side inequality:

$$\limsup_{n\to\infty}\frac{1}{\sqrt{n}}\mathcal{C}_{n,\,r}\left(L^2_a,\,H^2\right)\leq\sqrt{\frac{1+r}{1-r}},$$

which becomes obvious since

$$\frac{1}{\sqrt{n}}\mathcal{C}_{n,\,r}\left(L^2_a,\,H^2\right)\leq\frac{1}{\sqrt{n}}\sqrt{\mathcal{C}_{n,\,r}\left(H^2,\,H^2\right)},.$$

and

$$\frac{1}{\sqrt{n}}\sqrt{\mathcal{C}_{n,\,r}\left(H^2,\,H^2\right)}\to\sqrt{\frac{1+r}{1-r}},$$

as n tends to infinity, see [Z1] p. 2.

Step 2. We now prove the left-hand-side inequality:

$$\liminf_{n\to\infty}\frac{1}{\sqrt{n}}\mathcal{C}_{n,\,r}\left(L^2_a,\,H^2\right)\geq\sqrt{\frac{1+r}{1-r}}.$$

More precisely, we show that

$$\liminf_{n\to\infty}\frac{1}{\sqrt{n}}\left\Vert D\right\Vert_{\left(K_{b_r^n},\,\Vert\cdot\Vert_{L^2_a}\right)\to H^2}\geq\sqrt{\frac{1+r}{1-r}}.$$

Let $f\in K_{b_r^n}$. Then,

$$f'=\left(f,\,e_1\right)_{H^2}\frac{r}{(1-rz)}e_1+\sum_{k=2}^{n}(k-1)\left(f,\,e_k\right)_{H^2}\frac{b_r'}{b_r}e_k+r\sum_{k=2}^{n}\left(f,\,e_k\right)_{H^2}\frac{1}{(1-rz)}e_k=$$

$$=r\sum_{k=1}^{n}\left(f,\,e_k\right)_{H^2}\frac{1}{(1-rz)}e_k+\frac{1-r^2}{(1-rz)(z-r)}\sum_{k=2}^{n}(k-1)\left(f,\,e_k\right)_{H^2}e_k=$$

$$=\frac{r\left(1-r^2\right)^{\frac{1}{2}}}{\left(1-rz\right)^2}\sum_{k=1}^{n}\left(f,\,e_k\right)_{H^2}b_r^{k-1}+\frac{\left(1-r^2\right)^{\frac{3}{2}}}{(1-rz)^2(z-r)}\sum_{k=2}^{n}(k-1)\left(f,\,e_k\right)_{H^2}b_r^{k-1}=$$

$$=-b_r'\left[\frac{r}{\left(1-r^2\right)^{\frac{1}{2}}}\sum_{k=1}^{n}\left(f,\,e_k\right)_{H^2}b_r^{k-1}+\frac{\left(1-r^2\right)^{\frac{1}{2}}}{z-r}\sum_{k=2}^{n}(k-1)\left(f,\,e_k\right)_{H^2}b_r^{k-1}\right].$$

Now using the change of variables $v=b_r(u)$, we get

$$\Vert f'\Vert^2_{L^2_a}=\int_{\mathbb{D}}\left|b_r'(u)\right|^2\left|\frac{r}{\left(1-r^2\right)^{\frac{1}{2}}}\sum_{k=1}^{n}\left(f,\,e_k\right)_{H^2}b_r^{k-1}+\frac{\left(1-r^2\right)^{\frac{1}{2}}}{u-r}\sum_{k=2}^{n}(k-1)\left(f,\,e_k\right)_{H^2}b_r^{k-1}\right|^2\mathrm{d}u=$$

$$=\int_{\mathbb{D}}\left|\frac{r}{\left(1-r^2\right)^{\frac{1}{2}}}\sum_{k=1}^{n}\left(f,\,e_k\right)_{H^2}v^{k-1}+\frac{\left(1-r^2\right)^{\frac{1}{2}}}{b_r(v)-r}\sum_{k=2}^{n}(k-1)\left(f,\,e_k\right)_{H^2}v^{k-1}\right|^2\mathrm{d}v.$$

Now, $b_r - r = \frac{r-z-r(1-rz)}{1-rz} = \frac{z(r^2-1)}{1-rz}$, which gives

$$\|f'\|_{L^2_a}^2 = \int_{\mathbb{D}} \left| \frac{r}{(1-r^2)^{\frac{1}{2}}} \sum_{k=1}^{n} (f,\, e_k)_{H^2}\, v^{k-1} + \frac{(1-r^2)^{\frac{1}{2}}}{v(r^2-1)} (1-rv) \sum_{k=2}^{n} (k-1)\, (f,\, e_k)_{H^2}\, v^{k-1} \right|^2 \mathrm{d}v =$$

$$= \frac{1}{1-r^2} \int_{\mathbb{D}} \left| r \sum_{k=1}^{n} (f,\, e_k)_{H^2}\, v^{k-1} - (1-rv) \sum_{k=2}^{n} (k-1)\, (f,\, e_k)_{H^2}\, v^{k-2} \right|^2 \mathrm{d}v =$$

$$= \frac{1}{1-r^2} \int_{\mathbb{D}} \left| r \sum_{k=0}^{n-1} (f,\, e_{k+1})_{H^2}\, v^{k} - (1-rv) \sum_{k=0}^{n-2} (k+1)\, (f,\, e_{k+2})_{H^2}\, v^{k} \right|^2 \mathrm{d}v.$$

Thus,

$$\frac{1}{\|f\|_{H^2}\sqrt{n(1+r)}} \left[\left\| (1-rv) \sum_{k=0}^{n-2} (k+1)\, (f,\, e_{k+2})_{H^2}\, v^k \right\|_{L^2_a} + \left\| r \sum_{k=0}^{n-1} (f,\, e_{k+1})_{H^2}\, v^k \right\|_{L^2_a} \right] \geq \tag{16}$$

$$\geq \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L^2_a}}{\|f\|_{H^2}} \geq$$

$$\geq \frac{1}{\|f\|_{H^2}\sqrt{n(1+r)}} \left[\left\| (1-rv) \sum_{k=0}^{n-2} (k+1)\, (f,\, e_{k+2})_{H^2}\, v^k \right\|_{L^2_a} - \left\| r \sum_{k=0}^{n-1} (f,\, e_{k+1})_{H^2}\, v^k \right\|_{L^2_a} \right].$$

Now,

$$(1-rv) \sum_{k=0}^{n-2} (k+1)\, (f,\, e_{k+2})_{H^2}\, v^k =$$

$$= \sum_{k=0}^{n-2} (k+1)\, (f,\, e_{k+2})_{H^2}\, v^k - r \sum_{k=0}^{n-2} (k+1)\, (f,\, e_{k+2})_{H^2}\, v^{k+1} =$$

$$= \sum_{k=0}^{n-2} (k+1)\, (f,\, e_{k+2})_{H^2}\, v^k - r \sum_{k=1}^{n-1} k\, (f,\, e_{k+1})_{H^2}\, v^{k} =$$

$$= (f,\, e_2)_{H^2} + 2\, (f,\, e_3)_{H^2}\, v + \sum_{k=2}^{n-2} \left[(k+1)\, (f,\, e_{k+2})_{H^2} - rk\, (f,\, e_{k+1})_{H^2} \right] v^k +$$

$$- r \left[(f,\, e_2)_{H^2}\, v + (n-1)\, (f,\, e_n)_{H^2}\, v^{n-1} \right] =$$

$$= (f,\, e_2)_{H^2} + \left[(f,\, e_3)_{H^2} - r\, (f,\, e_2)_{H^2} \right] v + \sum_{k=2}^{n-2} \left[(k+1)\, (f,\, e_{k+2})_{H^2} - rk\, (f,\, e_{k+1})_{H^2} \right] v^k +$$

$$- r(n-1)\, (f,\, e_n)_{H^2}\, v^{n-1},$$

which gives

$$\left\|(1-rv)\sum_{k=0}^{n-2}(k+1)\,(f,\,e_{k+2})_{H^2}\,v^k\right\|_{L^2_a}^2 = \tag{17}$$

$$= |(f,\,e_2)_{H^2}|^2+\frac{1}{2}\,|(f,\,e_3)_{H^2}-r\,(f,\,e_2)_{H^2}|^2+$$

$$+\frac{1}{n}r^4(n-1)^2\,|(f,\,e_n)_{H^2}|^2+\sum_{k=2}^{n-2}\left|(f,\,e_{k+2})_{H^2}-\frac{rk}{k+1}\,(f,\,e_{k+1})_{H^2}\right|^2.$$

On the other hand,

$$\left\|r\sum_{k=0}^{n-1}(f,\,e_{k+1})_{H^2}\,v^k\right\|_{L^2_a}\leq r\left(\sum_{k=0}^{n-1}\frac{1}{k+1}\,|(f,\,e_{k+1})_{H^2}|^2\right)^{1/2}\leq r\,\|f\|_{H^2}\,, \tag{18}$$

Now, let $s=(s_n)$ be a sequence of even integers such that

$$\lim_{n\to\infty}s_n=\infty \text{ and } s_n=o(n) \text{ as } n\to\infty.$$

Then we consider the following function f in $K_{b_r^n}$:

$$f=\sum_{k=0}^{s+2}(-1)^k e_{n-k}.$$

Applying (17) with such an f, we get

$$\left\|(1-rv)\sum_{k=0}^{n-2}(k+1)\,(f,\,e_{k+2})_{H^2}\,v^k\right\|_{L^2_a}^2 =$$

$$= r^4\frac{(n-1)^2}{n}+$$

$$+\sum_{l=2}^{n-2}(n-l+1)\left|(f,\,e_{n-l+2})_{H^2}-\frac{r(n-l)}{n-l+1}\,(f,\,e_{n-l+1})_{H^2}\right|^2,$$

setting the change of index $l=n-k$ in the last sum. This finally gives

$$\left\|(1-rv)\sum_{k=0}^{n-2}(k+1)\,(f,\,e_{k+2})_{H^2}\,v^k\right\|_{L^2_a}^2 =$$

$$= r^4 \frac{(n-1)^2}{n} + \sum_{l=2}^{s+1} (n-l+1) \left| 1 + \frac{r(n-l)}{n-l+1} \right|^2 =$$

$$= r^4 \frac{(n-1)^2}{n} + \sum_{l=2}^{s+1} (n-l+1) \left[1 + r \left(1 - \frac{1}{n-l+1} \right) \right]^2 ,$$

and

$$\left\| (1-rv) \sum_{k=0}^{n-2} (k+1) \, (f, \, e_{k+2})_{H^2} \, v^k \right\|_{L^2_a}^2 \geq$$

$$\geq r^4 \frac{(n-1)^2}{n} + (s+1-2+1)(n-(s+1)+1) \left[1 + r \left(1 - \frac{1}{n-(s+1)+1} \right) \right]^2 =$$

$$= r^4 \frac{(n-1)^2}{n} + s(n-s) \left[1 + r \left(1 - \frac{1}{n-s} \right) \right]^2 .$$

In particular,

$$\left\| (1-rv) \sum_{k=0}^{n-2} (k+1) \, (f, \, e_{k+2})_{H^2} \, v^k \right\|_{L^2_a}^2 \geq s(n-s) \left[1 + r \left(1 - \frac{1}{n-s} \right) \right]^2 .$$

Now, since $\|f\|^2_{H^2} = s_n + 3$, we get

$$\liminf_{n\to\infty} \frac{1}{n \, \|f\|^2_{H^2}} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) \, (f, \, e_{k+2})_{H^2} \, v^k \right\|_2^2 \geq$$

$$\geq \liminf_{n\to\infty} \frac{1}{n \, \|f\|^2_{H^2}} \, \|f\|^2_{H^2} \left(n - \|f\|^2_{H^2} \right) \left[1 + r \left(1 - \frac{1}{n-s} \right) \right]^2 =$$

$$= \lim_{n\to\infty} \left(1 - \frac{s_n}{n} \right) \left[1 + r \left(1 - \frac{1}{n-s} \right) \right]^2 = (1+r)^2 .$$

On the other hand, applying (18) with this f, we obtain

$$\lim_{n\to\infty} \frac{1}{\sqrt{n} \, \|f\|_{H^2}} \left\| r \sum_{k=0}^{n-1} (f, \, e_{k+1})_{H^2} \, v^k \right\|_{L^2_a} = 0.$$

Thus, we can conclude passing after to the limit as n tends to $+\infty$ in (16), that

$$\liminf_{n\to\infty} \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L^2_a}}{\|f\|_{H^2}} = \frac{1}{\sqrt{1+r}} \liminf_{n\to\infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| (1-rv) \sum_{k=0}^{n-2} (k+1) \, (f, \, e_{k+2})_{H^2} \, v^k \right\|_{L^2_a} \geq$$

$$\geq \frac{1+r}{\sqrt{1+r}} = \sqrt{1+r},$$

and

$$\liminf_{n\to\infty} \sqrt{\frac{1-r}{n}} \|D\|_{K_{b_r^n}\to H^2} \geq \liminf_{n\to\infty} \sqrt{\frac{1-r}{n}} \frac{\|f'\|_{L_a^2}}{\|f\|_{H^2}} \geq \sqrt{1+r}.$$

Step 3. Conclusion. Using both **Step 1** and **Step 2**, we get

$$\limsup_{n\to\infty} \sqrt{\frac{1-r}{n}} \mathcal{C}_{n,\,r}\left(L_a^2,\ H^2\right) = \liminf_{n\to\infty} \sqrt{\frac{1-r}{n}} \mathcal{C}_{n,\,r}\left(L_a^2,\ H^2\right) = 1+r,$$

which means that the sequence $\left(\frac{1}{\sqrt{n}}\mathcal{C}_{n,\,r}\left(L_a^2,\ H^2\right)\right)_{n\geq 1}$ is convergent and

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \mathcal{C}_{n,\,r}\left(L_a^2,\ H^2\right) = \sqrt{\frac{1+r}{1-r}}.$$

$\square$

Proof of Theorem B.

Proofs of inequality (8) and of the right-hand side inequality of (10). Let σ be a sequence in $\mathbb{D}$, and $B = B_\sigma$ the finite Blaschke product corresponding to σ. If $f \in H^2$, we use the same function g as in [Z3] which satisfies $g_{|\sigma} = f_{|\sigma}$. More precisely, let $g = P_B f \in K_B$ (see Definitions 2, 3 and Remark 4 above for the definitions of K_B and P_B). Then $g - f \in BH^2$ and using the definition of $\mathcal{C}_{n,\,r}\left(L_a^2,\ H^2\right)$,

$$\|g'\|_{L_a^2}^2 \leq \left(\mathcal{C}_{n,\,r}\left(L_a^2,\ H^2\right)\right)^2 \|g\|_{H^2}^2 .$$

Now applying the identity (2) to g we get

$$\|g\|_{B_{2,\,2}^{\frac{1}{2}}}^2 \leq \left[\left(\mathcal{C}_{n,\,r}\left(L_a^2,\ H^2\right)\right)^2 + 1\right] \|g\|_{H^2}^2 .$$

Using the fact that $\|g\|_{H^2} = \|P_B f\|_{H^2} \leq \|f\|_{H^2}$, we finally get

$$\|g\|_{B_{2,\,2}^{\frac{1}{2}}} \leq \left[\left(\mathcal{C}_{n,\,r}\left(L_a^2,\ H^2\right)\right)^2 + 1\right]^{\frac{1}{2}} \|f\|_{H^2} ,$$

and as a result,

$$I\left(\sigma,\ H^2,\ B_{2,\,2}^{\frac{1}{2}}\right) \leq \left[\left(\mathcal{C}_{n,\,r}\left(L_a^2,\ H^2\right)\right)^2 + 1\right]^{\frac{1}{2}} .$$

It remains to apply the right-hand side inequality of (4) in Theorem A to prove the right-hand side one of (10).

Proof of inequality (9). 1) We use the same test function

$$f = \sum_{k=0}^{n-1} (1-|\lambda|^2)^{\frac{1}{2}} b_\lambda^k \left(1-\overline{\lambda}z\right)^{-1},$$

as the one used in the proof of [Z3, Theorem B] (the lower bound, page 11 of [Z3]). f being the sum of n elements of H^2 which are an orthonormal family known as Malmquist's basis (associated with $\sigma_{n,\lambda} = \underbrace{\{\lambda, \lambda, ..., \lambda\}}_{n}$, see Remark 4 above or [N1, p.117]) , we have $\|f\|_{H^2}^2 = n$.

2) Since the spaces H^2 and $B_{2,2}^{\frac{1}{2}}$ are rotation invariant, we have $I\left(\sigma_{n,\lambda}, H^2, B_{2,2}^{\frac{1}{2}}\right) = I\left(\sigma_{n,\mu}, H^2, B_{2,2}^{\frac{1}{2}}\right)$ for every λ, μ with $|\lambda| = |\mu| = r$. Let $\lambda = -r$. To get a lower estimate for $\|f\|_{B_{2,2}^{\frac{1}{2}}/b_\lambda^n B_{2,2}^{\frac{1}{2}}}$ consider g such that $f - g \in b_\lambda^n \mathrm{Hol}(\mathbb{D})$, i.e. such that $f \circ b_\lambda - g \circ b_\lambda \in z^n \mathrm{Hol}(\mathbb{D})$.

3) First, we notice that

$$\|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 = \left\|(g \circ b_\lambda)'\right\|_{L_a^2}^2 + \|g \circ b_\lambda\|_{H^2}^2 = \|b_\lambda.(g' \circ b_\lambda)\|_{L_a^2}^2 + \|g \circ b_\lambda\|_{H^2}^2 =$$

$$= \int_{\mathbb{D}} |b_\lambda(u)|^2 \left|g'(b_\lambda(u))\right|^2 du + \|g \circ b_\lambda\|_{H^2}^2 = \int_{\mathbb{D}} \left|g'(w)\right|^2 dw + \|g \circ b_\lambda\|_{H^2}^2 ,$$

using the changing of variable $w = b_\lambda(u)$. We get

$$\|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 = \|g'\|_{L_a^2}^2 + \|g \circ b_\lambda\|_{H^2}^2 = \|g\|_{B_{2,2}^{\frac{1}{2}}}^2 + \|g \circ b_\lambda\|_{H^2}^2 - \|g\|_{H^2}^2 ,$$

and

$$\|g\|_{B_{2,2}^{\frac{1}{2}}}^2 = \|g\|_{H^2}^2 + \|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2 =$$

$$\geq \|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2 .$$

Now, we notice that

$$f \circ b_\lambda = \sum_{k=0}^{n-1} z^k \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\overline{\lambda} b_\lambda(z)} = \left(1-|\lambda|^2\right)^{-\frac{1}{2}} \left(1 + (1-\overline{\lambda}) \sum_{k=1}^{n-1} z^k - \overline{\lambda} z^n\right) =$$

$$= (1-r^2)^{-\frac{1}{2}} \left(1 + (1+r) \sum_{k=1}^{n-1} z^k + r z^n\right) .$$

4) Next,

$$\|g \circ b_\lambda\|_{B_{2,2}^{\frac{1}{2}}}^2 - \|g \circ b_\lambda\|_{H^2}^2 = \sum_{k\geq 1} k \left|\widehat{g \circ b_\lambda}(k)\right|^2 \geq$$

$$\geq \sum_{k=1}^{n-1} k \left|\widehat{g \circ b_\lambda}(k)\right|^2 = \sum_{k=1}^{n-1} k \left|\widehat{f \circ b_\lambda}(k)\right|^2 ,$$

since $\widehat{g\circ b_\lambda}(k)=\widehat{f\circ b_\lambda}(k)$, $\forall\, k\in[0,\,n-1]$. This gives

$$\|g\circ b_\lambda\|^2_{B^{\frac{1}{2}}_{2,\,2}}-\|g\circ b_\lambda\|^2_{H^2}\geq\frac{1}{1-r^2}\left((1+r)^2\sum_{k=1}^{n-1}k\right)=$$

$$=\frac{(1+r)^2}{1-r^2}\frac{n(n-1)}{2}=\frac{1+r}{1-r}\frac{n(n-1)}{2}=\frac{1+r}{1-r}\frac{(n-1)}{2}\,\|f\|^2_{H^2}\,,$$

for all $n\geq 2$ since $\|f\|^2_{H^2}=n$. Finally,

$$\|g\|^2_{B^{\frac{1}{2}}_{2,\,2}}\geq\frac{n}{1-r}\frac{1+r}{2}\left(1-\frac{1}{n}\right)\|f\|^2_{H^2}\,.$$

In particular,

$$\mathcal{I}_{n,\,r}\left(H^2,\,B^{\frac{1}{2}}_{2,2}\right)\geq\sqrt{\frac{n}{1-r}}\left[\frac{1+r}{2}\left(1-\frac{1}{n}\right)\right]^{\frac{1}{2}}.$$

□

Some comments.

a. Extension of Theorem A to spaces $B^s_{2,\,2}$, $s\geq 0$. Using the techniques developped in the proof of our Theorem A (combined with complex interpolation (between Banach spaces) and a reasoning by induction), it is possible both to precise the sharp numerical constant $c_{2,\,s}$ in K. Dyakonov's result (3) (mentioned above in paragraph d. of the Introduction) and to prove the asymptotic sharpness (at least for $s\in\mathbb{N}\cup\frac{1}{2}\mathbb{N}$) of the right-hand side inequality of (3). In the same spirit, we would obtain that there exists a limit:

$$\lim_{n\to\infty}\frac{\mathcal{C}_{n,\,r}\left(B^{s-1}_{2,\,2},\,H^2\right)}{n^s}=\left(\frac{1+r}{1-r}\right)^s. \tag{19}$$

Our Theorem A corresponds to the case $s=\frac{1}{2}$.

b. Extension of Theorem B to spaces $B^s_{2,\,2}$, $s\geq 0$**.** The proof of the upper bound in our Theorem B can be extended so as to give an upper (asymptotic) estimate of the interpolation constant $\mathcal{I}_{n,\,r}\left(H^2,\,B^s_{2,2}\right)$, $s\geq 0$. More precisely, applying K. Dyakonov's result (3) (mentioned above in paragraph d. of the Introduction) we get

$$\mathcal{I}_{n,\,r}\left(H^2,\,B^s_{2,2}\right)\leq\tilde{c}_s\left(\frac{n}{1-r}\right)^s,\text{ with }\tilde{c}_s\asymp c_{2,\,s}, \tag{20}$$

where $c_{2,\,s}$ is defined in (3) and precised in (19). Looking at the above comment 1, $\tilde{c}_s\asymp(1+r)^s$ for sufficiently large values of n. Our Theorem B corresponds again to the case $s=\frac{1}{2}$. In this Theorem B, we prove the sharpness of the right-hand side inequality in (20) for $s=\frac{1}{2}$. However, for the general case $s\geq 0$, the asymptotic sharpness of $\left(\frac{n}{1-r}\right)^s$ as $r\to 1^-$ and $n\to\infty$ is less obvious. Indeed, the key of the proof (for the sharpness) is based on the property that the Dirichlet norm (the one of $B_{2,2}^{1/2}$) is "nearly" invariant composing by an elementary Blaschke factor b_λ, as this is the case for the H^∞ norm. A conjecture given by N. K. Nikolski is the following:

$$\mathcal{I}_{n,r}\left(H^2,\, B_{2,2}^s\right) \asymp \begin{cases} \frac{n^s}{\sqrt{1-r}} \text{ if } s \geq \frac{1}{2} \\ \left(\frac{n}{1-r}\right)^s \text{ if } 0 \leq s \leq \frac{1}{2} \end{cases}, \tag{21}$$

and is due to the position of the spaces $B_{2,2}^s$, $s \geq 0$ with respect to the algebra H^∞.

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